UNDERSTANDING RATIONAL TANGLES

THE MATHEMATICS BEHIND RATIONAL TANGLES AND THE RATIONAL TANGLE DANCE (PLUS A NEW DANCE!)

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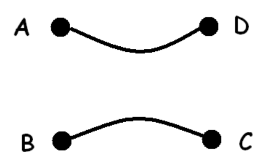
1. INTRODUCTION

In 1967, mathematician John H. Conway presented seminal results on the classification of knots through a study of "rational tangles." In order to explain some of key ideas behind the theory, he developed an activity, the "rational tangle dance," that four people holding the ends of two lengths of rope, and a fifth person watching on, can perform. The magic of this activity has captivated many over the decades, and the rational tangle dance is a now considered a favourite in many math circle groups.

Here's the dance:

Four people stand at positions A, B, C and D and hold two ropes in the initial configuration shown.



These folk may "dance" with these ropes by performing just two moves, multiple times, in any order they like:

- Rotate as a group 90° counterclockwise. (Thus the person in position A moves to position B, the person in position B to position C, and so on.) Call this move "ROTATE".

- Folks in positions D and C swap places with D holding the rope up and over C. Call this move a "SWAP."

Thus a dance tangles the ropes.

After the dancers have danced for a while, a towel (actually a garbage bag is better as it is lighter) is placed over the tangle, and a fifth person, who has been watching the dance all along, yells out another series of instructions: ROTATE, SWAP, SWAP, ROTATE, for example, and then STOP. The towel is then removed and the dancers are flabbergasted to see their tangle completely untangled.

How does the fifth person do this?

This trick really does seem surprising as it seems that swapping in the same direction and rotating in the same direction could never possibly "undo" previous moves of these same types.

Here's what the fifth person does:

Starting with the number zero, she records a new number each time the dancers perform a move. For each SWAP she adds one to the number in hand and each time they ROTATE she inverts the number and multiplies by negative one:

SWAP: $x \mapsto x+1$

ROTATE: $x \mapsto -\frac{1}{x}$

[The Dancers will realize that starting with a ROTATE leads to nothing fruitful. We can assume the dance starts with a SWAP and so we are not in a position of dividing by zero.]

Once the dancers finish, the fifth person has a fraction in mind, a single "tangle number."

To untangle the dance, she follows the following instructions:

If the tangle number is negative tell the dancers to SWAP and add 1 to the tangle number to obtain a new number.

If the tangle number is positive, tell the dancers to ROTATE and take the negative reciprocal of the tangle number to obtain a new one.

If the tangle number is zero, STOP. The ropes are untangled.

By repeating these instructions, the fifth person will, for certain, reach the value zero and at this point the ropes will be untangled to the surprise of all.

The reason why these instructions work are equally as mysterious as the why the tangle dance itself untangles!

At this point one can play the tangle dance game, and have students play it too to practice their manipulation of fractions. And without the deep understanding of the mathematics of the tangles themselves, there is deep play to be had with the manipulation of fractions. See **Tom Davis's** super paper **CONWAY'S RATIONAL TANGLES** available on the web (google "Tom Davis Rational Tangle") for a rich discussion of this.

For a long time I have been intrigued by this dance trick, wondering what the mathematics hidden behind it really was. The goal of this document is to explain, in as straightforward terms possible, the mathematics behind rational tangles.

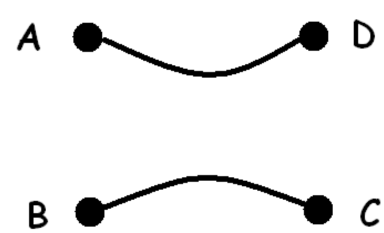
We begin, in part 1, by explaining what a tangle is and begin developing some notation of for them on paper.

In part 2 we attempt to link some basic operations on tangles with matching operations in arithmetic, and explain why the name *rational tangle* is appropriate: each tangle can be matched with a rational number.

Part 3 proves the key mathematical result: *IF TWO TANGLES HAVE THE SAME FRACTION ASSOCIATED WITH THEM, THEN THEY ARE SAME TANGLE.* Since we associate the number zero with the untangled set of ropes, any tangle that we later produce again with tangle number zero must then be physically manipulable to the untangled state.

This result explains establishes why the tangle dance works, which we discuss in part 4. It also allows us to invent dances of our own. For example, here is a new dance:

Again four people stand at positions A, B, C and D and hold two ropes in the initial configuration as shown.



This time folk "dance" following these two moves, multiple times, in any order:

- Folks in positions D and C swap places with D holding the rope up and over C. Call this a "Side Swap" and denote it S.
- Folks in positions B and C swap places, with B holding the rope up and over C. Call this a "Front Swap" and denote it F.

NOTICE: All lifted ropes move to the diagonal position C.

Maybe members of the audience can yell out Ss and Fs and the dancers follow suit.

After the dancers have danced for a while the ropes will be quite tangled. A fifth person, who has been watching the dance all along, then stands up and asks the dancers to perform a single 90° rotation. (They can choose which direction: clockwise or counterclockwise). He turns his back to the dancers and calls out a series of Ss and Fs (watch out, the person in position A who had no action in the dance is now taking part in the dance as she is now in position B) and the dancers move appropriately.

After the fifth person has made all her calls the ropes, magically, are untangled!

How does the fifth person do this?

We explain this dance too.

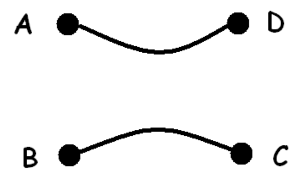
In part 5 we discuss some curious connections to related topics in mathematics, and the Appendix completes the mathematical picture by proving the converse of the key result: *If one tangle can be physically manipulated into a second tangle, then those two tangles have the same tangle number.* This shows that Conway's matching of tangles with rational tangles provides a complete classification of these objects.

LEVEL OF MATHEMATICS:
The mathematics described in these notes, in and of itself, is not too technically advanced: basic knowledge of high-school arithmetic and algebra is all that is needed. (There is one proof completed by induction.) However, working through this document does require *work*! The sequence of ideas developed is sophisticated and some may find it quite taxing. Patience, perseverance, and persistence are required to properly understand this document. All is explained carefully and all is accessible. Nonetheless, take time in your reading and enjoy the thinking journey rational tangles offer.

TIP! *Buy two one-foot lengths of light-weight chain from a hardware store (chain a little heavier than that used for necklaces). One can lay these chains on a table top and experiment with tangles very easily and effectively using them!*

1. THE TANGLE SET-UP

Four positions labeled A, B, C and D are marked on the ground at the vertices of a square and four people stand at these positions, one person at each label. Folks in positions A and D hold a rope between them, as do the folk at positions B and C. These ropes are untangled and this is our starting position.

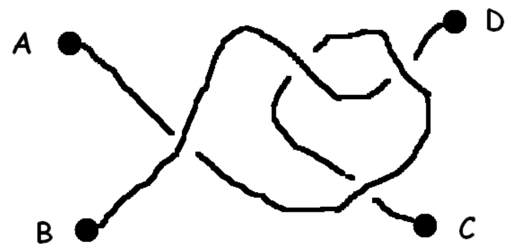


Folk will move and tangle the ropes the ropes in the process. There are EIGHT moves they are allowed to make, each given by having two adjacent folk swap places with one ducking under the other's end of the rope. For example, folks in positions A and B could swap places with A lifting the rope up over B, or, if the ropes are more tangled than shown in the above starting configuration, folks in positions B and C could swap places with B ducking under C's end of the rope.

Note: Having A and D folk swap places, or B and C folk swap places as a first move has no effect on the initial configuration.

As an example, here is the tangle that results from the series of moves:

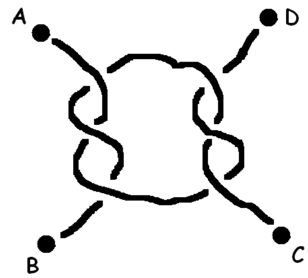
i) Folks in positions A and B swap, twice, with B lifting up over A.
ii) Folks in positions B and C swap with C lifting up over B.
iii) The (new) folks in positions A and B swap with A lifting up over B.



Definition: Any tangle of ropes that results from performing a finite number of such swap moves, in some order, is called a *rational tangle.*

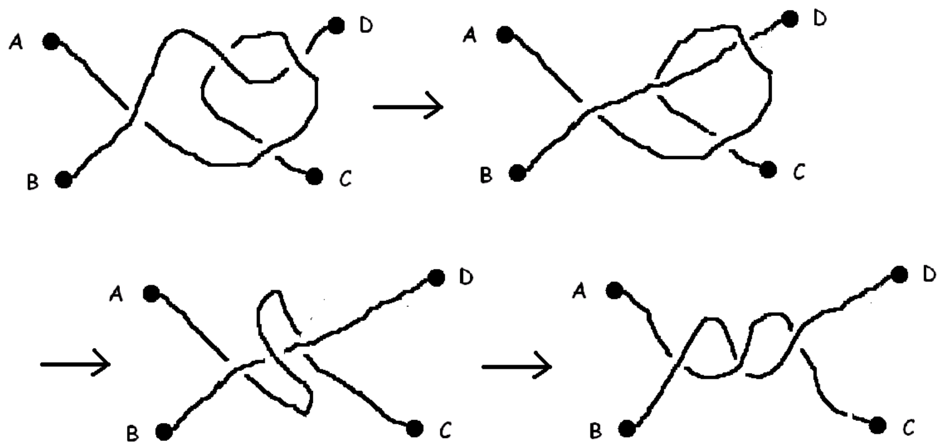
This name comes from that fact that these tangles are intimately connected with rational numbers, as we shall see.

NOTE: Not every tangle of ropes is produced this way. For example, this tangle is not a rational tangle.

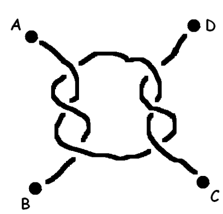


Loose Definition: Two configurations of ropes, with the folk standing at positions A, B, C and D holding on tight and never letting go of their ends, are said to be *isotopic* if it is possible for a fifth person to come along and maneuver the ropes of one configuration, staying within the space at the center of the square formed by the four people, and transform it into an exact copy of the second configuration.

For example, the tangle in the previous tangle is isotopic to the one produced by folks in positions A and B swapping three times in a row:



COMMENT: Even with maneuvering this configuration cannot be transformed into a tangle that is a rational tangle.

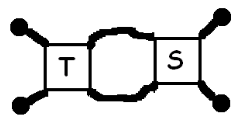


This seems intuitively obvious, but is hard to justify with solid mathematics! (Care to try?)

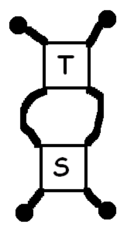
But this figure does look like two rational tangles "added together" side-to-side and suggests a basic definition for combining tangles.

Definition: If T and S denote two configurations of ropes (not necessarily rational tangles)

then their *horizontal sum,* denoted $T \underset{horiz}{+} S$, and *vertical sum*, $T \underset{vert}{+} S$, are the configurations of ropes implied by the diagrams:



Horizontal Sum



Vertical Sum

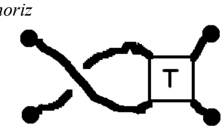
NOTE: $T \underset{horiz}{+} S$ and $S \underset{horiz}{+} T$ need not be isotopic. (Similarly for the vertical sum.) These "addition" operations are generally not commutative.

When two adjacent people swap places they are, in effect, performing a horizontal or vertical sum on the tangle already present. Precisely, if we denote a single crossing oriented this way  as $[1]$, and a single crossing with reverse orientation  as $[-1]$, then the eight moves the four folk can perform to add to a tangle T are:

$$[\pm 1] \underset{horiz}{+} T \quad T \underset{horiz}{+} [\pm 1]$$

$$[\pm 1] \underset{vert}{+} T \quad T \underset{vert}{+} [\pm 1]$$

For example, the action of folk in positions A and B swapping places, with B lifting the rope up and over A, gives $[1] \underset{horiz}{+} T$.



$[1] \underset{horiz}{+} T$

A rational tangle is thus any configuration of ropes that results from applying any finite sequence of the "moves"

$$[\pm 1] \underset{horiz}{+} \ldots \quad \text{or} \quad \ldots \underset{horiz}{+} [\pm 1] \quad \text{or} \quad [\pm 1] \underset{vert}{+} \ldots \quad \text{or} \quad \ldots \underset{vert}{+} [\pm 1]$$

to the initial starting configuration.

THEOREM 1: If T is a rational tangle, then

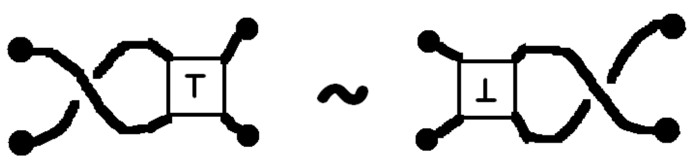
$$[1] \underset{horiz}{+} T \sim T \underset{horiz}{+} [1],$$

where $\sim$ means "is isotopic to." We also have:

$$[-1] \underset{horiz}{+} T \sim T \underset{horiz}{+} [-1]$$
$$[1] \underset{vert}{+} T \sim T \underset{vert}{+} [1]$$
$$[-1] \underset{vert}{+} T \sim T \underset{vert}{+} [-1]$$

Proof: We prove the first relation. The remaining three are proved in the same way.

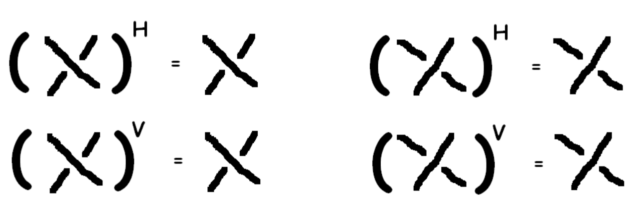
We certainly have:



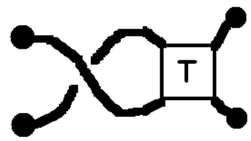
That is: $[1] \underset{horiz}{+} T \sim T^H \underset{horiz}{+} [1]$ where T^H is the reflection of the tangle T about a horizontal axis. For the sake of completeness let's denote by T^V the reflection of a tangle T about a vertical axis. The proof will follows once we establish:

LEMMA 2: For any rational tangle T we have: $T^H \sim T$ and $T^V \sim T$.

To prove this lemma, note that it is true for rational tangles with just one crossing:



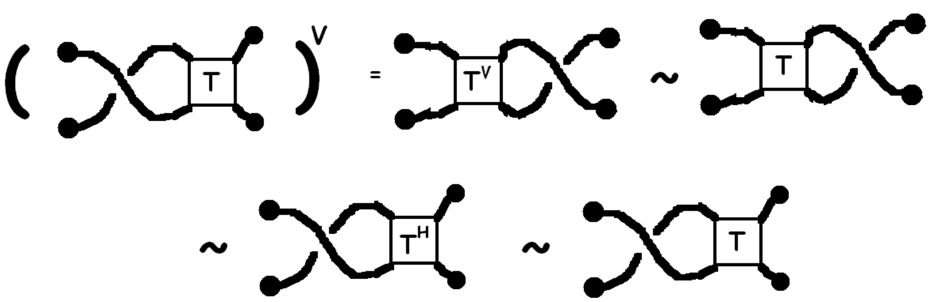
Suppose the result is true for a rational tangle with N crossings and consider a tangle with one more crossing. It could one of eight possible forms. Let's consider the form:



where T is a tangle with N crossings. (The remaining seven forms are studied in the same way as what follows.) By the induction hypothesis we know $T^H \sim T$ and $T^V \sim T$.

Now:

and



So the result is true for a tangle with $N+1$ crossings as well. By induction, the lemma holds, and hence the theorem holds. □

COROLLARY 3:

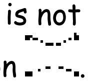
a) The 180° rotation of a rational tangle is ~ to the original tangle.
b) A 90° clockwise rotation of a rational tangle is ~ to a 90° counterclockwise rotation of the tangle.

Proof: A 180° is induced by a horizontal flip followed by a vertical flip. We have $\left(T^H\right)^V \sim T^V \sim T$. Part b) follows from a) since these differ by a 180° rotation. □

Comment: Let "R" denote the 90° rotation of a rational tangle. It is clear that:

$$[1]^{R}=[-1] \text{ and } [-1]^{R}=[1]$$

and an induction argument on the number of crossings shows that if T is a rational tangle, then T^{R} is as well. Corollary 3 shows that we need not specify whether R is clockwise or counterclockwise rotation.

Comment within the comment: It is <u>not</u> clear yet how to best regard the 90° rotation of the initial configuration .

Theorem 1 shows that we need never have folks in positions A and B swap places (their action is equivalent to having C and D swap places) and we need never have A and D swap positions. Thus a rational tangle can be constructed as a finite sequence from just four possible moves.

$$T \underset{horiz}{+} [\pm 1] \qquad\qquad T \underset{vert}{+} [\pm 1]$$

NOTATION: It seems appropriate to give $[1] \underset{horiz}{+} [1]$ the name $[2]$:

[2]

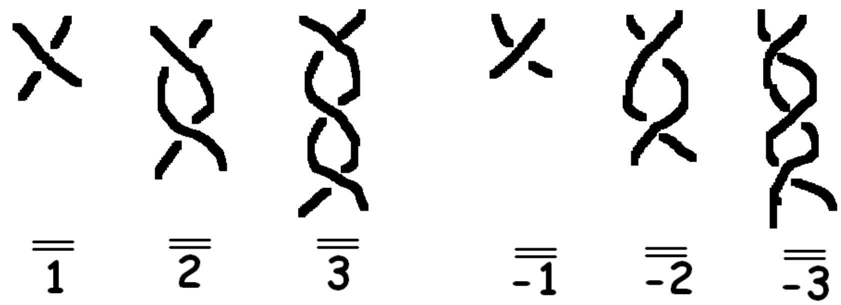
and $[-1] \underset{horiz}{+} [-1] \underset{horiz}{+} [-1]$ the name $[-3]$, and so on.

[-3]

So maybe we should call the initial state $[0]$:

[0]

And it seems natural to give a notation for vertical sums. If left and right square brackets denote the horizontal sums, it would be good to use above and below square brackets for the vertical sums. Alas, math software does not offer this symbol for typing, so we will use:



Notice:

$$[1] = \overline{\overline{1}} \text{ and } [-1] = \overline{\overline{-1}} \text{ and } \overline{\overline{1}}^{R} = \overline{\overline{-1}} \text{ etc}$$

Comment: Let's give  the name $\overline{\overline{0}}$.

LEMMA 4:
If a and b are integers, then $[a] \underset{horiz}{+} [b] = [a+b]$ and $\overline{\overline{a}} \underset{vert}{+} \overline{\overline{b}} = \overline{\overline{a+b}}$.

Proof: This is clear. □

More generally:

LEMMA 5: If a and b are integers and T is a rational tangle, then
$\left(T \underset{horiz}{+} [a]\right) \underset{horiz}{+} [b] = T \underset{horiz}{+} [a+b]$ and $\left(T \underset{vert}{+} \overline{\overline{a}}\right) \underset{vert}{+} \overline{\overline{b}} = T \underset{vert}{+} \overline{\overline{a+b}}$.

It follows from these two lemmas that since a rational tangle is constructed by performing a finite sequence of operations $\ldots \underset{horiz}{+} [\pm 1]$ and $\ldots \underset{vert}{+} \overline{\overline{\pm 1}}$, we can view any rational tangle as being constructed as an alternating sum of the form:

$$\left(\left(\left([a] \underset{vert}{+} \overline{\overline{b}}\right) \underset{horiz}{+} [c]\right) \underset{vert}{+} \overline{\overline{d}}\right) \underset{horiz}{+} \ldots$$

for some non-zero integers $a, b, c, d, \ldots$.

COMMENT: From now on we won't write the parentheses in expressions like these. But it is to be understood that the order of operations is to read from left to right. This is important as these operations are not associative. For instance, $\left([a] \underset{vert}{+} \overline{\overline{b}}\right) \underset{horiz}{+} [c]$ is generally not isotopic to $[a] \underset{vert}{+} \left(\overline{\overline{b}} \underset{horiz}{+} [c]\right)$.

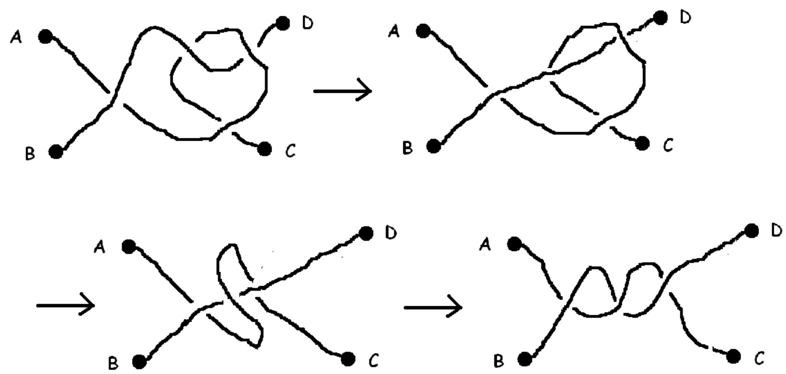
So to summarize we can say:

Definition: A *rational tangle* is a configuration of ropes constructed from any finite sequence of operations

$$[a] \underset{vert}{+} \overline{\overline{b}} \underset{horiz}{+} [c] \underset{vert}{+} \overline{\overline{d}} \underset{horiz}{+} \ldots$$

for non-zero integers $a, b, c, d, \ldots$.

WARNING: Two different expressions can yield the same configuration of ropes. For example, the diagrams:



show

$$[3] \sim [1] \underset{horiz}{+} \left([2] \underset{vert}{+} \overline{\overline{-1}}\right) \sim [2] \underset{vert}{+} \overline{\overline{-1}} \underset{horiz}{+} [1]$$

Comment: From now on we will write just "$+$" for $\underset{horiz}{+}$ and for $\underset{vert}{+}$. The context of which sum to use is always clear. (Adding $[x]$ must be done via a horizontal sum and adding $\overline{\overline{x}}$ via a vertical sum.) A rational tangle will now be written:

$$[a] + \overline{\overline{b}} + [c] + \overline{\overline{d}} + \ldots$$

2. THE ARITHMETIC OF RATIONAL TANGLES

It is a standard practice in mathematics to try to associate with each object in a class of geometric objects (knots, tangles, surfaces,...) a number so that if two objects are physically equivalent in the geometry, they have the same number. This then allows one to prove that two objects are different: If they do not have the same number, then they are not physically the same.

So for tangles, we'd like a function:

$$f : \text{Rational Tangles} \rightarrow \text{Numbers}$$

At the very least, we'd like this function to respect the basic equivalence $\left(T+[a]\right)+[b]=T+[a+b]$ and $\left(T+\overline{\overline{a}}\right)+\overline{\overline{b}}=T+\overline{\overline{a+b}}$. So we need:

$$f\left(\left(T+[a]\right)+[b]\right)=f\left(T+[a+b]\right)$$
$$f\left(\left(T+\overline{\overline{a}}\right)+\overline{\overline{b}}\right)=f\left(T+\overline{\overline{a+b}}\right)$$

That is, f has to be an operation on numbers that respects the notion "if you do a and then do b, you obtain the same result as doing $a+b$" but in two different ways: one for an operation of horizontal sums and one for the operation of vertical sums.

There are certainly a number of operations in arithmetic for which "a then b is the same as $a+b$" holds.

EXAMPLE 1: **Addition**:
Start with x. Adding a and then adding b gives the same result as adding $a+b$

$$x \mapsto \quad x+a \quad \mapsto \quad x+a+b=x+\left(a+b\right)$$

EXAMPLE 2: **Subtraction:**
Start with x. Subtracting a and then subtracting b gives the same result as subtracting $a+b$

$$x \mapsto \quad x-a \quad \mapsto \quad x-a-b=x-\left(a+b\right)$$

EXAMPLE 3: **Powers of a variable.**
Let θ be a variable and let "doing a" correspond to multiplying by θ^a.
Start with x. Doing a and then doing b gives the same result as doing $a+b$

$$x \mapsto \quad x \cdot \theta^a \quad \mapsto \quad x \cdot \theta^a \cdot \theta^b = x \cdot \theta^{a+b}$$

This example will take us to the algebra of polynomials in the variable θ rather than the algebra of numbers. But it is a possibility we could play with if we like.

EXAMPLE 4: **From Continued Fractions**
Those familiar with continued fractions might think of the following operation. "Doing a" means "invert, add a, and invert again." Weird!

Start with x. Doing a and then doing b does give the same result as doing $a+b$

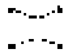
$$x \mapsto \quad \frac{1}{a+\frac{1}{x}} \quad \mapsto \quad \frac{1}{b+\frac{1}{\left(\frac{1}{a+\frac{1}{x}}\right)}} = \frac{1}{(a+b)+\frac{1}{x}}$$

There are probably more examples.

Since we have two distinct operations, horizontal sum and vertical sum, we should use two distinct examples from arithmetic. Subtraction is the reverse of addition, so we shouldn't use examples 1 and 2. We shan't go into polynomial algebra (though it would be fun to try) so we should avoid 3. Let's use examples 1 and 4.

And to give us a starting point, it seems natural to assign the number zero to the starting configuration .

Definition: Define a function $f : \textit{Rational Tangles} \rightarrow \textit{Numbers}$ by:

$f($  $) = 0$

$$f\left(T + [a\}\right) = f\left(T\right) + a$$

$$f\left(T + \overline{\overline{a}}\right) = \cfrac{1}{a + \cfrac{1}{f\left(T\right)}}$$

Let's see how this function works:

$f\left([1]\right) = f($  $+[1]) = f($  $) + 1 = 0 + 1 = 1$

$$f\left([2]\right) = f\left([1]+[1]\right) = f\left([1]\right) + 1 = 1 + 1 = 2$$

$$f\left([3]\right) = f\left([2]+[1]\right) = f\left([2]\right) + 1 = 2 + 1 = 3$$

In general:

$f\left(\left[a\right]\right) = a$ for any non-negative integer a.

We can also show

$f\left(\left[a\right]\right) = a$ for any negative integer a.

We also have:

$$f\left(\overline{\overline{1}}\right) = f\left([1]\right) = 1$$

$$f\left(\overline{\overline{2}}\right) = f\left([1] + \overline{\overline{1}}\right) = \cfrac{1}{1 + \cfrac{1}{f\left([1]\right)}} = \frac{1}{1+1} = \frac{1}{2}$$

$$f\left(\overline{\overline{3}}\right)=f\left([1]+\overline{\overline{2}}\right)=\frac{1}{2+\frac{1}{f([1])}}=\frac{1}{3}$$

And in general:

$$f\left(\overline{\overline{a}}\right)=\frac{1}{a}$$ for positive (and negative) integers a.

Comment: It seems that $f\left(\overline{\overline{0}}\right)="\frac{1}{0}"$. Shall we call this "$\infty$"?

We have:

$f([a])=a$ and $f\left(\overline{\overline{a}}\right)=\frac{1}{a}$ for positive and negative integers a.
$f([0])=0$ and $f\left(\overline{\overline{0}}\right)=\infty$.

So what is the number associated to a general rational tangle $[a]+\overline{\overline{b}}+[c]+\overline{\overline{d}}+\ldots$?

Let's do this in stages.

$$f([a])=a$$

$$f\left([a]+\overline{\overline{b}}\right)=\frac{1}{b+\frac{1}{f([a])}}=\frac{1}{b+\frac{1}{a}}=0+\frac{1}{b+\frac{1}{a}}$$

$$f\left([a]+\overline{\overline{b}}+[c]\right)=f\left([a]+\overline{\overline{b}}\right)+c=c+\frac{1}{b+\frac{1}{a}}$$

$$f\left([a]+\overline{\overline{b}}+[c]+\overline{\overline{d}}\right)=\frac{1}{d+\frac{1}{f\left([a]+\overline{\overline{b}}+[c]\right)}}=0+\frac{1}{d+\frac{1}{c+\frac{1}{b+\frac{1}{a}}}}$$

etc.

Associated with a rational tangle is a continued fraction. Using the square bracket notation for continued fractions we have:

$$f\left([a]+\overline{\overline{b}}+[c]+\overline{\overline{d}}+\ldots+[z]\right)=[z,y,x,\ldots,c,b,a]=z+\cfrac{1}{y+\cfrac{1}{x+\cdots\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{a}}}}}$$

and

$$f\left([a]+\overline{\overline{b}}+[c]+\overline{\overline{d}}+\ldots+\overline{\overline{y}}\right)=f\left([a]+\overline{\overline{b}}+[c]+\overline{\overline{d}}+\ldots+\overline{\overline{y}}+[0]\right)$$

$$=[0,y,x,\ldots,c,b,a]=0+\cfrac{1}{y+\cfrac{1}{x+\cdots\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{a}}}}}$$

Comment: A tangle that begins $\overline{\overline{a}}+[b]+\overline{\overline{c}}+\cdots$ can be thought of as $[1]+\overline{\overline{a-1}}+[b]+\overline{\overline{c}}+\cdots$ and one can check that $f\left(\overline{\overline{a}}+[b]+\overline{\overline{c}}+\cdots\right)$ does equal $f\left([1]+\overline{\overline{a-1}}+[b]+\overline{\overline{c}}+\cdots\right)$. (Also think about the case $a=1$.) So we can assume that all our tangles do indeed begin with a term of the form $[a]$. Also, we can also assume all our tangles end with a term of the form $[z]$. (Use $[0]$ if need be.)

Every rational tangle can be thought of as sum of an <u>odd</u> number of terms:

$$[a]+\overline{\overline{b}}+[c]+\cdots+\overline{\overline{y}}+[z]$$

with a non-zero. We have $f\left([a]+\overline{\overline{b}}+[c]+\cdots+\overline{\overline{y}}+[z]\right)=[z,y,\ldots,c,b,a]$.

TWO GEOMETRIC EFFECTS ON THE ARITHMETIC:

1. MIRROR IMAGE.

If one looks at a tangle $T=[a]+\overline{\overline{b}}+[c]+\cdots+\overline{\overline{y}}+[z]$ in the mirror, then all the over crossings become under crossings, and vice versa. It T^M is the mirror image of T, then we have:

$$T^M=[-a]+\overline{\overline{-b}}+[-c]+\cdots+\overline{\overline{-y}}+[-z]$$

Arithmetic shows $-z+\cfrac{1}{-y+\cfrac{1}{-x+\cdots\cfrac{1}{-b+\cfrac{1}{-a}}}}$ is the negative of $z+\cfrac{1}{y+\cfrac{1}{x+\cdots\cfrac{1}{b+\cfrac{1}{a}}}}$.

We have:

$$\boxed{f\left(T^M\right)=-f(T)}$$

2. 90 DEGREE ROTATION.

If $T=[a]+\overline{\overline{b}}+[c]+\cdots+\overline{\overline{y}}+[z]$, then it is not hard to convince oneself that

$$T^R=\overline{\overline{-a}}+[-b]+\overline{\overline{-c}}\cdots+\overline{\overline{-z}}$$

and this can be rewritten $T^R=[-1]+\overline{\overline{-a+1}}+[-b]+\overline{\overline{-c}}\cdots+\overline{\overline{-z}}+[0]$. So

$$f\left(T^R\right)=0+\cfrac{1}{-z+\cfrac{1}{-y+\cdots\cfrac{1}{-c+\cfrac{1}{-b+\cfrac{1}{-a+1+\cfrac{1}{-1}}}}}}=\cfrac{1}{-z+\cfrac{1}{-y+\cdots\cfrac{1}{-c+\cfrac{1}{-b+\cfrac{1}{-a}}}}}=\frac{1}{-f(T)}$$

We have:

$$\boxed{f\left(T^R\right)=-\frac{1}{f(T)}}$$

THE STANDARD FORM OF A RATIONAL TANGLE

So far we have:

Every rational tangle can be thought of as sum of an odd number of terms:

$$[a]+\overline{\overline{b}}+[c]+\cdots+\overline{\overline{y}}+[z]$$

with a non-zero.

We have $f\left([a]+\overline{\overline{b}}+[c]+\cdots+\overline{\overline{y}}+[z]\right)=[z,y,\ldots,c,b,a]$.

We now prove that every rational tangle can be represented as a sum of terms $[a]+\overline{\overline{b}}+[c]+\cdots+\overline{\overline{y}}+[z]$ with each integer (except possibly the final z, which could be zero) having the same sign: all positive or all negative.

LEMMA 6: For a rational tangle T we have:

$$T^R+\overline{\overline{1}}\sim T+\overline{\overline{-1}}+[1].$$

Also:

$$T^R+\overline{\overline{-1}}\sim T+\overline{\overline{1}}+[-1]$$

$$T^R+[1]\sim T+[-1]+\overline{\overline{1}}$$

$$T^R+[-1]\sim T+[1]+\overline{\overline{-1}}$$

Proof: We prove the first. The others are established similarly. A sequence of diagrams does the trick:

□

So lemma 6 tells us that if a and b are positive integers, then

$$\begin{aligned} T + \overline{\overline{-b}} + [a] &= T + \overline{\overline{-b+1}} + \overline{\overline{-1}} + [1] + [a-1] \\ &\sim \left(T + \overline{\overline{-b+1}}\right)^R + \overline{\overline{1}} + [a-1] \\ &= T^R + [b-1] + \overline{\overline{1}} + [a-1] \end{aligned}$$

(And there are three other variations of this identity.)

This result says that if two consecutive terms in the expression $[a] + \overline{\overline{b}} + [c] + \cdots + \overline{\overline{y}} + [z]$ have opposite signs, then they can be replaced by three terms of the same sign, but the portion to the left of that expression, $[a] + \overline{\overline{b}} + \ldots$, changes to $\overline{\overline{-a}} + [-b] + \ldots$, as dictated by the effect of a 90° rotation.

So … Start with the right most term in $[a] + \overline{\overline{b}} + [c] + \cdots + \overline{\overline{y}} + [z]$ and move towards the left until you find the first term of opposite sign to z (or to y if $z = 0$). Replace that pair with three terms the same sign as z, and many of the signs to the left of this will change.

Now read to the left some more, until you find the next pair of opposite sign. Keep doing this until you have moved all the way to the left.

The final result is an expression for the tangle with all terms the same sign.

COMMENT: We have to be careful with potential zero terms. If $a = 1$, then $T + \overline{\overline{-b}} + [a] + \overline{\overline{w}} \sim T^R + [b-1] + \overline{\overline{1}} + [a-1] + \overline{\overline{w}}$ equals $T^R + [b-1] + \overline{\overline{1}} + [0] + \overline{\overline{w}}$, which equals $T^R + [b-1] + \overline{\overline{w+1}}$ and the result we hope for will still occur. One should carefully check the details of this.

This argument establishes:

THEOREM 7: Every rational tangle can be thought of as sum of an odd number of terms:

$$[a] + \overline{\overline{b}} + [c] + \cdots + \overline{\overline{y}} + [z]$$

with a non-zero and all terms of the same sign.

We have: $f\left([a] + \overline{\overline{b}} + [c] + \cdots + \overline{\overline{y}} + [z]\right) = [z, y, \ldots, c, b, a]$.

This is called a *standard form* of the rational tangle.

3. THE BIG RESULT – WELL HALF OF IT

We are now in to position to prove:

THEOREM 8: If two rational tangles S and T have the same rational number associated to each of them, $f(S) = f(T)$, then the two tangles are isotopic.

In particular, if $f(T) = 0$ for some rational tangle T, then that tangle can be untangled to the initial state.

The converse of theorem 8 is also true: *If two tangles are isotopic, then they have the same rational number associated to each of them.* It takes a bit more work to establish this.

We have seen that every rational tangle T can be regarded as (is isotopic to) a tangle of standard form $[a] + \overline{\overline{b}} + [c] + \cdots + \overline{\overline{y}} + [z]$ with all terms the same sign (except z could be zero). T has the rational number $z + \cfrac{1}{y + \cfrac{1}{x + \cdots \cfrac{1}{b + \cfrac{1}{a}}}}$ associated with it. Let's assume all the terms are positive (otherwise, work with T^M) and call the value of this continued fraction ω. This number is a positive rational, greater than or equal to 1 if $z \neq 0$ and less than 1 if $z = 0$. Recall there are an odd number of terms in this expression (allowing $z = 0$ to possibly being one of those terms).

It is a standard result in continued fraction theory (and left as an exercise for you) that a positive rational ω can be expressed as a continued fraction with positive integers in only one way – except for the variation at the tail given by $b + \frac{1}{a} = b + \cfrac{1}{a - 1 + \frac{1}{1}}$. If we insist that the expression have an odd number of terms, then we know whether to use $b + \frac{1}{a}$ or $b + \cfrac{1}{a - 1 + \frac{1}{1}}$ at its tail. Thus:

There is only one expression for ω as a continued fraction with an odd number of terms.

This means that whenever a rational tangle S leads to the number $f(S)=\omega$, there is only one possible standard form for S.

Thus we have:

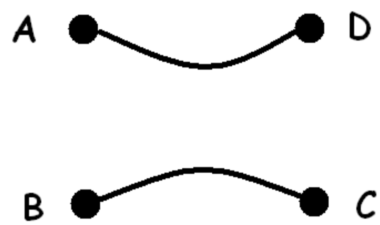
> T is isotopic to the tangle given by its standard form.
> S is isotopic to the tangle given by the same standard form.

Thus $S \sim T$.

4. FUN AND GAMES

Let's now discuss the rational tangle dance. Here again is its description.

Four people stand at positions A, B, C and D and hold two ropes in the initial configuration $[0]$.



These folk may "dance" with these ropes by performing just two moves, multiple times, in any order they like:

- R : rotate as a group counterclockwise 90°. Call this "ROTATE".
- $+[1]$: Folks in positions D and C swap places with D holding the rope up and over C. Call this "SWAP."

Thus a dance tangles the ropes.

After the dancers have danced for a while, a towel (actually a garbage bag is better as it is lighter) is placed over the tangle, and a fifth person, who has been watching the dance all along, yells out another series of instructions: ROTATE, SWAP, SWAP, ROTATE, for example, and then STOP. The towel is then removed and the dancers are flabbergasted to see their tangle completely untangled.

How does the fifth person do this?

This trick really does seem surprising as it seems that swapping in the same direction and rotating in the same direction could never possibly "undo" previous moves of these same types.

The Answer:
Of course the fifth person, as she watched the dance, was keeping track of the tangle number being produced. We start with the number $f([0]) = 0$. The dancers will realize that starting with any set of ROTATEs has no meaningful effect, so we can assume the dance effectively starts with a SWAP.

Of course any swap move takes the tangle number we currently have and adds 1 to it, and every rotate move computes its negative reciprocal:

SWAP:
$$T \to T + [1]$$
$$f(T) \mapsto f(T) + 1$$

ROTATE:
$$T \to T^R$$
$$f(T) \mapsto -\frac{1}{f(T)}$$

These moves produce a tangle number $\pm\frac{p}{q}$, a positive or negative rational.

The instructions the fifth person offers are given by the following algorithm:

If the tangle number is negative tell the dancers to SWAP and add 1 to the tangle number to obtain a new number.

If the tangle number is positive, tell the dancers to ROTATE and take the negative reciprocal of the tangle number to obtain a new one.

If the tangle number is zero, STOP. By theorem 8, the tangle must be isotopic to the non-tangle.

Following this algorithm, the fifth person is sure to eventually yell STOP. Here's why:

If $\pm\frac{p}{q}$ is a negative integer, then the algorithm will add one to this number repeatedly until the value zero is reached.

If $\pm\frac{p}{q}$ is a negative non-integer, namely $-\frac{p}{q}$ with p and q positive integers, then the algorithm will add one to this number until it reaches a positive value between 0 and 1.

If $\pm\frac{p}{q}$ is a positive value greater than 1, then the algorithm will take the negative reciprocal of this value and then add one to it. It will produce a positive rational between 0 and 1.

If $\pm\frac{p}{q}$ equals 1, then the algorithm will produce the value zero in two steps.

So the only case left to consider is the case that $\frac{p}{q}$ is a positive rational between 0 and 1. What will the algorithm do? It will take the negative reciprocal and add 1 repeated to obtain a new rational between 0 and 1, or if we are lucky, to obtain zero.

Thus $\frac{p}{q}$ is transformed to $-\frac{q}{p}+n$ where $0 \le -\frac{q}{p}+n < 1$.

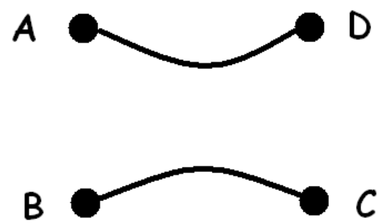
To see what n is, write $q = kp + r$ for some remainder $0 \le r < p$. Then $-\frac{q}{p} = -k + \frac{r}{p}$ and we see that n must be k, and we now have the number $-\frac{q}{p} + n = \frac{r}{p}$.

Thus the algorithm transforms the fraction $\frac{q}{p}$ between 0 and 1 to a new fraction $\frac{r}{q}$ between 0 and 1 with smaller denominator. Eventually the denominator 0 must appear and we have reached the untangled state!

At last … This completely explains the Conway's Rational Tangle dance!

ANOTHER DANCE:

Again four people stand at positions A, B, C and D and hold two ropes in the initial configuration $[0]$.



This time folk "dance" following these two moves, multiple times, in any order:

- $+[1]$: Folks in positions D and C swap places with D holding the rope up and over C. Call this a "Side Swap" and denote it S.
- $+\overline{\overline{1}}$: Folks in positions B and C swap places, with B holding the rope up and over C. Call this a "Front Swap" and denote it F.

NOTICE: All lifted ropes move to the diagonal position C.

Maybe members of the audience can yell out Ss and Fs and the dancers follow suit.

After the dancers have danced for a while the ropes will be quite tangled. A fifth person, who has been watching the dance all along, then stands up and asks the dancers to perform a single 90° rotation. (They can choose which direction: clockwise or counterclockwise). He turns his back to the dancers and calls out a series of Ss and Fs (watch out, the person in position A who had no action in the dance is now taking part in the dance as she is now in position B) and the dancers move appropriately.

After the fifth person has made all her calls the ropes, magically, are untangled!

How does the fifth person do this?

The Answer:
The fifth person recorded the sequence of Ss and Fs the dancers performed:

SSSSFFFSSFSSSFSSSFSSSSSFFF….

After the rotation was performed she simply reads out the same sequence backwards, but saying "F" for each S, and "S" for each F. For example, if dancers performed the dance SSFFFS the fifth person, after the rotation, instructs FSSSFF.

This works because any dance performed has the form:

$$T+[a]+\overline{\overline{b}}+[c]+\overline{\overline{d}}+\cdots+[z]$$

(with $a,b,c,...$ positive integers with z possibly zero). After the rotation, the tangle is of the form:

$$\left(T+[a]+\overline{\overline{b}}+[c]+\overline{\overline{d}}+\cdots+[z]\right)^R$$
$$=T^R+\overline{\overline{-a}}+[-b]+\overline{\overline{-c}}+[-d]+\cdots+\overline{\overline{-z}}$$

and the instructions read out by the fifth person give

$$T^R+\overline{\overline{-a}}+[-b]+\overline{\overline{-c}}+[-d]+\cdots+\overline{\overline{-z}}+[z]+\cdots+[d]+\overline{\overline{c}}+[b]+\overline{\overline{a}}=T^R$$

So if we started with $T=[0]$ the final result is the untangle state rotated 90°.

Comment: One can start with the four ropes tied tightly together in the middle with a simple knot. The obviates the need to start with a S move over an F move. As long as the dancers don't pull the ropes too tightly, the tangles they create will disappear and the same simple knot will appear at the end of the trick.

CHALLENGE: Make up your own magic dances.

5. THE STERN-BROCOT TREE, FORD CIRCLES and more!

To be written

APPENDIX: Proof that $S \sim T$ implies $f(S) = f(T)$.

To be written